Temasek Junior College
2020 JC 2 Prelim Exam H2 Mathematics Paper 1

1 Given that $A=\sqrt{1+\frac{1}{x^2}}$, and that the rate of decrease of A is k times the rate of increase of x when $x = k$, find k, where k is a positive real number. [4]

2 **(a)** The region R is bounded by the curve $y = x\mathrm{e}^{-x}$, the line $x = n$ and the x-axis, where n is a positive real number.

(i) Find, in terms of n, the volume of the solid generated, V, when R is rotated 2π radians about the x-axis. [3]

(ii) Hence find $\lim_{n\to\infty} V$, giving your answer in exact form. [1]

[You may assume that as $n\to\infty$, $n\mathrm{e}^{-2n}\to 0$ and $n^2\mathrm{e}^{-2n}\to 0$.]

(b) Describe a sequence of two transformations such that the graph of $y = a^x$ is mapped onto the graph of $y = a^{b-x}$, where a and b are real numbers. [2]

3 On the same diagram, sketch the graphs of $y = |ax - 1|$, where $a > 1$, and $y = x^2 - 1$. Give, in terms of a, the coordinates of the points where the curves meet the axes and the x-coordinates of their intersection points. [4]

(i) Hence find the solution set of the inequality $|ax - 1| > x^2 - 1$. [1]

(ii) Deduce the solution set of the inequality $|a^{x+1} - 1| > a^{2x} - 1$. [2]

4 **(a)** The complex number w is such that $w^3 = -8\mathrm{i}$. Given that one possible value of w is $2\mathrm{i}$, use a **non-calculator method** to find the other values of w. Give your answers in the form $a+b\mathrm{i}$, where a and b are exact values. [3]

(b) Find the principal argument of the complex number $q=-1+\mathrm{i}\sqrt{3}$. Hence, or otherwise, show that there is no integer value of n for which the real part of $\frac{q^n}{q^*}$ is zero. [4]

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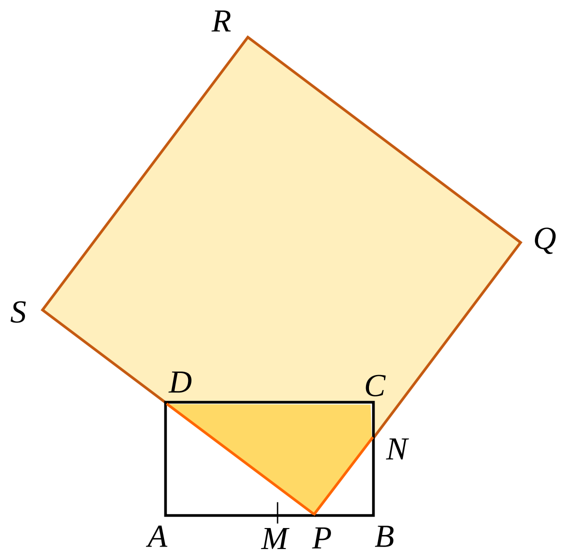


The diagram (not drawn to scale) shows a piece of rectangular paper *ABCD* with *AB* = 10 cm, *BC* = 5 cm and *M* is the midpoint of *AB*. A piece of square translucent cellophane paper *PQRS*, with side 15 cm, is overlaid on *ABCD* such that *P* lies on *MB*, *SDP* is collinear, and *PQ* intersects *BC* at *N*.

(i) Denoting the length of *AP* by *x* cm, show that $BN = \dfrac{x(10-x)}{5}$. [2]

(ii) Using an analytical method, find the exact maximum area of the region in *ABCD* overlaid by *PQRS*. [6]

6 **(i)** Given that $T_r = \dfrac{r^2}{2^r}$, show that $T_r - T_{r+1} = \dfrac{(r-1)^2}{2^{r+1}} - \dfrac{1}{2^r}$. [2]

(ii) Hence, find $\displaystyle\sum_{r=1}^{N}\left(\frac{(r-1)^2}{2^{r+1}} - \frac{1}{2^r}\right)$ giving your answer in the form $\dfrac{1}{2} - \mathrm{f}(N)$. [2]

(iii) Show that $\displaystyle\sum_{r=1}^{N}\frac{(r-1)^2}{2^{r+1}} = \frac{3}{2} - \frac{(N+1)^2}{2^{N+1}} - \frac{1}{2^N}$. [2]

(iv) Deduce an expression for $\displaystyle\sum_{r=1}^{N-1}\frac{r^2}{2^r}$ in terms of *N*. [3]

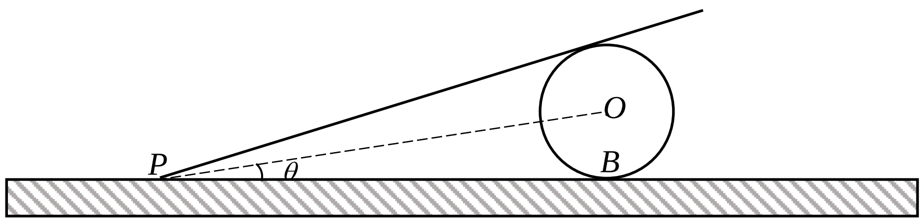
7 In the latest simulation game, Speedy Cheetah Family Simulator, cheetahs need to hunt down and eat gazelles to maintain their health and survive in the simulated Savana forest. The game simulates how a cheetah chases a gazelle. The animals' movements are modelled by a series of leaps in the same direction in a straight line. The cheetah's first leap is 6 m and each subsequent leap is shorter than its preceding leap by 10 cm. The gazelle's first leap is 3 m and each subsequent leap is 98% of its preceding leap. Assume both the cheetah and the gazelle start leaping at the same moment and the time taken for each leap is the same.

(i) Find, in terms of n, the total distance covered by the gazelle after n leaps. [2]

(ii) Find, in terms of n, the total distance covered by the cheetah after n leaps. If the gazelle is 32 m away from the cheetah at the start of the chase, find the number of leaps taken by the cheetah to catch the gazelle. [4]

(iii) Find the number of leaps the cheetah takes before coming to a complete stop. Deduce the shortest distance (to the nearest tenth of a metre) that the gazelle must be from the cheetah at the start of the chase in order to survive the chase. [4]

8 The diagram below (not drawn to scale) shows the vertical cross section of a cylindrical drum with centre O and radius 5 cm. The drum is fixed at a point B on the horizontal ground and a rectangular metal sheet leans on it.



Initially, the metal sheet touches the horizontal ground at point P such that PB = 50 cm and $\angle OPB = \theta$ radians. The metal sheet is moved in the direction PB so that P is h cm closer to B and $\angle OPB$ is increased by α radians, where α is small.

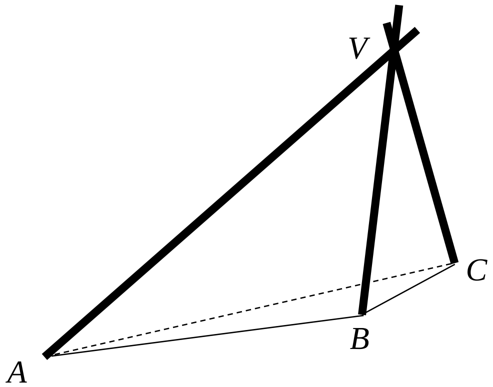
(i) Using a suitable compound angle formula in MF26, show that

$$h \approx \frac{505\alpha}{1+10\alpha}.$$ [4]

(ii) Hence express h as a series in α, up to and including the term in α^3. [3]

(iii) Given that P is moving towards B at a rate of 0.1 cm s^{-1}, find the rate of change of α when h = 1. [4]

9 A camper built a tripod tent using 3 poles and 2 canvases as shown in the diagram below. The 3 poles rest on **sloping ground** at the points *A*, *B*, and *C*, and are fastened together at the point *V*. With respect to the origin (not shown in the diagram), *A*, *B* and *C* have position vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $7\mathbf{i} + 4\mathbf{j}$ and $8\mathbf{i} + 12\mathbf{j} + 2\mathbf{k}$ respectively, where $\mathbf{k}$ is perpendicular to the horizontal. You may assume that the sloping ground is a plane.



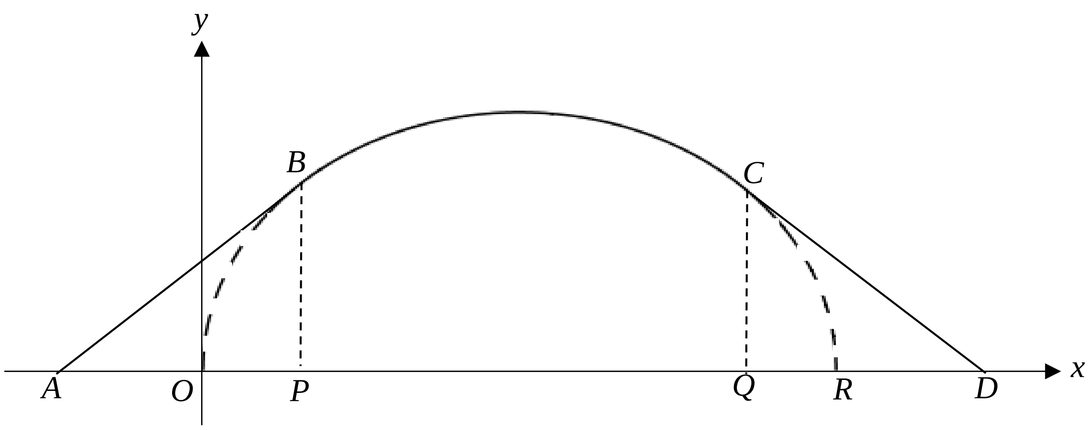
(i) Find a vector equation, in parametric form, of the sloping ground. [2]

(ii) The vectors $\overrightarrow{AV}$ and $\overrightarrow{CV}$ are parallel to $4\mathbf{i} + \alpha\mathbf{j} + 10\mathbf{k}$ and $-2\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}$ respectively. Find, in either order, the value of α and the position vector of *V*. [4]

(iii) Find the acute angle between the pole *CV* and the sloping ground. [3]

(iv) The camper attaches a small torchlight to the top of the tent at *V*. The torchlight was not fastened properly and dropped vertically down to the ground. Assuming the lowest end of the torch is at *V*, find the distance travelled by the torchlight from *V* to the ground. [3]

10 The figure below shows a **symmetrical** design for a suspension bridge arch *ABCD*.



The arch design consists of the curved part *BC* and the straight lines *AB* and *CD*. The curved part *BC* is part of the curve *OBCR* with parametric equations given by

$$x = a(2t - \sin 2t), \quad y = a(1 - \cos 2t) \text{ for } 0 \le t \le \pi,$$

where *a* is a constant.

[Continued Q10]

(i) Find, in terms of a, the length of line segment OR and the height of the bridge at its highest point. [3]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cot t$. [2]

The straight lines AB and CD are tangents to the curve at B and C respectively and are inclined at 30 degrees to the horizontal.

(iii) Find the exact coordinates of B, in terms of a. [2]

(iv) Given $a = 6$, show that the area of the region bounded by the arch $ABCD$, the curve $OBCR$ and the x-axis is given by $h - k\int_0^{\alpha} (1-\cos 2t)^2 \, \mathrm{d}t$, where h, k and α are constants to be determined. Hence find this exact area. [6]

11 In a research laboratory, scientists carry out experiments to produce a new substance C. C is produced when two substances A and B are reacted together in a chemical reaction. At time t hours, the amount of C is x grams, and the amounts of A and B present are $(a - x)$ grams and $(b - x)$ grams respectively, where a and b are real constants. At the start of the experiment, there are no traces of C found. At any instant, the rate at which the amount of C increases is proportional to the product of the amount of A and the amount of B present at that instant.

(i) In an experiment, a scientist carries out the experiment with $b = a$.

(a) Obtain a differential equation relating x and t and solve for x in terms of t. [4]

(b) Sketch the graph of x against t. [1]

(c) Given that there is $\dfrac{1}{2}a$ grams of C produced 1 hour after the experiment started, find the time needed for $\dfrac{4}{5}a$ grams of C to be produced. [2]

(ii) Another scientist carries out the experiment with $b < a$.

(a) Obtain a differential equation relating x and t and solve for x in terms of t. [4]

(b) Using your answer in part **(ii)(a)**, find the amount of C produced after a long time. [2]